Relaxation dynamics in congested traffic

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(Received 5 July 2004; published 24 February 2005)

We investigate the relaxation dynamics in congested traffic when starting from the "megajam" initial condition (all cars standing in one big cluster of density 1) in the framework of the traffic model proposed by Nagel and Schreckenberg. On the one hand, a simple comparison of the time evolutions of some relevant traffic quantities shows that the slowest relaxing quantity is the density of "go and stop" cars rather than the average velocity of cars. On the other hand, we find that the relaxation time diverges in the form of a power law $\tau \approx \tau_0 p^{-\beta}$. A simple theoretical argument predicts that the exponent "β" is equal to 1. This prediction is consistent with the numerical result.

DOI: 10.1103/PhysRevE.71.026124 PACS number(s): 89.40.−a, 02.60.Cb, 05.65.+b, 45.70.Vn

Ever since Nagel and Schreckenberg (NS) elaborated their traffic flow model [1], cellular automata (CA) models have been used frequently in order to understand the complex dynamic behavior of traffic [2,3]. Among the interesting investigations of the traffic system is the study of the transition from free flow to congested states. In the deterministic case of the NS model (DNS), this transition, which is viewed as a second-order phase transition, occurs at the critical density $\rho_c = 1/(v_{\max}+1)$. Although several studies have been done in an effort to explain the nature of the transition for the nondeterministic NS model (NDNS), this question has been under debate until now [4–7]. Using a local-density analysis, Lübeck *et al.* [6] have investigated the characteristic fluctuations in a steady state and presented the phase diagram of the NDNS model. They suggested that the transition could be described as a phase transition of second order. In contrast, Gerwinski and Krug [8] argued that most features of the transition found in the DNS do not persist in the presence of noise. In addition, the authors found that the transition point ρ_c, which separates the free flow to congested traffic, can be obtained from simple considerations of the dissolution of the megajam, i.e., when all cars leave the megajam. Below ρ_c, the dissolution time of the megajam is very low, while far from ρ_c it become "infinite," i.e., the jam does not dissolve within the measurement time. Although there exist several studies of traffic jams in the literature [7–10], some open questions still remain. For example, what is the behavior of the relaxation time near the vanishing value of noise when starting from a megajam initial condition?

The NS model is a probabilistic CA of traffic flow on a one-lane roadway. It consists of N cars moving on a one-dimensional lattice of L cells with periodic boundary conditions (the number of vehicles N is conserved). Each cell is either empty or occupied by just one vehicle with velocity $v = 1, 2, \ldots, v_{\max}$. We denote by $x(k,t)$ and $v(k,t)$ the position and the velocity of the kth car at time t, respectively. The number of empty cells in front of the kth car is denoted by $g(k,t) = x(k+1,t) - x(k,t) - 1$ and is referred to hereafter as the gap. Space and time are discrete. At each discrete time step $t \rightarrow t+1$ the system update is performed in parallel for all cars according to the following four subrules.

R_1: acceleration: $v(k, t+\frac{1}{3}) \leftarrow \min(v(k,t)+1, v_{\max})$. R_2: slowing down (due to other cars): $v(k, t+\frac{2}{3}) \leftarrow \min(v(k, t+\frac{1}{3}), g(k,t))$. R_3: randomization (noise): $v(k,t+1) \leftarrow \max(v(k, t+\frac{2}{3})-1, 0)$ with probability p. R_4: motion: the car is moved forward according to its new velocity, $x(k,t+1) \leftarrow x(k,t) + v(k,t+1)$.

In order to characterize the behavior of the model, we perform global measurements on the system's lattice. These measurements are expressed as macroscopic quantities, defining the global density ρ, the space mean speed $\langle v \rangle(t)$ as

$$\rho = N/L; \quad \langle v \rangle(t) = \frac{1}{N}\sum_{k=1}^{N} v(k,t). \tag{1}$$

Let us introduce a new observable, m, called hereafter the density of "go and stop" cars,

$$m(t) = \frac{1}{N}\sum_{k=1}^{N} n_k(t)[1 - n_k(t+1)] \tag{2}$$

with $n_k = 0$ for stopped cars and $n_k = 1$ for moving cars.

Our results presented in Fig. 1 show that the relaxation time of the density of "go and stop" cars (m) is much higher than that of the average velocity ($\langle v \rangle$). Therefore, we stated that the slowest relaxing quantity is m rather than $\langle v \rangle$ [11]. Thus, the observable m is more suitable for the study of the relaxation behavior in the traffic systems. The "rule" published in some traffic works that the equilibration of the system is established by monitoring the time evolution of the average velocity of cars is a procedure that is clearly not valid in general.

In this section, we shall investigate the dynamics of the NS model in congested traffic when starting from the megajam initial condition. This is done by plotting the time evolutions of m and computing their relaxation times. Obviously, the time evolutions as well as the relaxation times should depend on both the density ρ and the randomization

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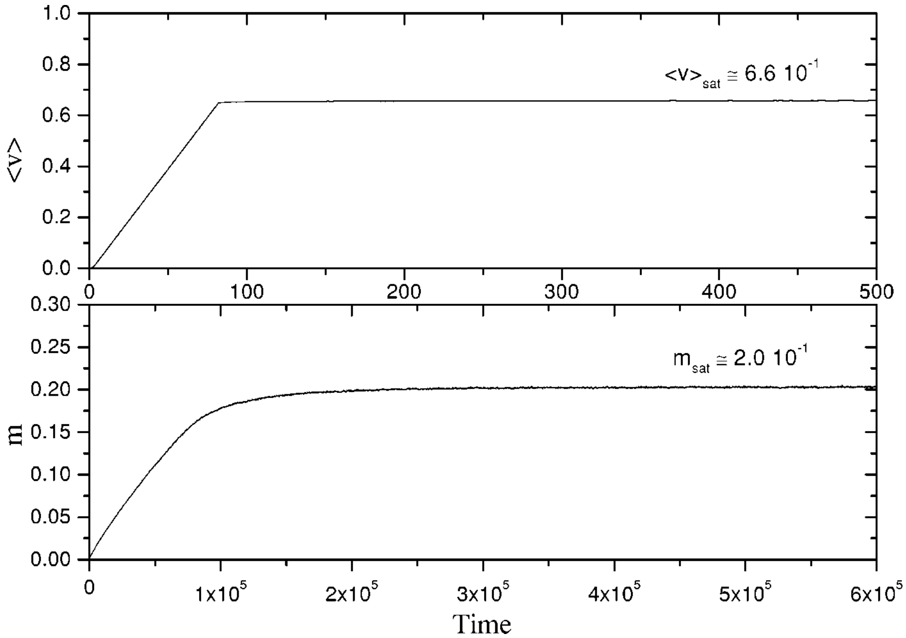

FIG. 1. Time evolution of the average velocity $\langle v\rangle$ and the density of "go and stop" cars m when starting from the megajam initial configuration ($\rho=0.6$, $p=0.005$, and $v_{\max}=5$). $\langle v\rangle_{sat}$ and m_{sat} are saturated values (equilibration) of $\langle v\rangle(t)$ and $m(t)$, respectively. The system size is $L=1000$.

p. In Figs. 2 and 3, we plotted the time evolutions of m, when starting from the megajam initial configuration, for several values of density and randomization. We find that the equilibration is delayed if the density of the megajam is increased (Fig. 2). Assuming the uniqueness of the stationary state of the NDNS model, the survival of the megajam should be equivalent to the occurrence of jams with infinite lifetimes for arbitrary initial condition [8]. Indeed, for high density, the megajam never dissolves but can dissociate into minijams after a certain long time. Next, we shall investigate the influence of randomization on the relaxation dynamics of the megajam. At this point, a remark about the two limits $p\rightarrow 0$ and $p\rightarrow 1$ is appropriate. At high density, one can expect that the relaxation time of the megajam diverges for $p\rightarrow 0$ as well as for $p\rightarrow 1$. The limit where $p\rightarrow 1$ is rather difficult to investigate numerically since the values of $\langle v\rangle$ as well as m become almost zero. In contrast, the limit $p\rightarrow 0$ is more accessible for the investigations, thus we shall restrict this study to this limit. For fixed high density, we plotted the time evolution of m, when starting from the megajam initial configuration, under various values of randomization (Fig. 3). This shows that the time equilibration is delayed when the randomization p is decreased towards zero. When $p\rightarrow 0$, the first car in the jam escapes easily since the restart probability is rather high. In addition, if the density is high, the escaping car reaches very quickly the back end of the jam because the velocity in the outflow region is maximal and the covered distance is rather small ($d=L-N$).

Assuming that the randomization p is the rate of transition for the dynamics of the NS model, all quantities (probabilities of different events) will involve the following combination only:

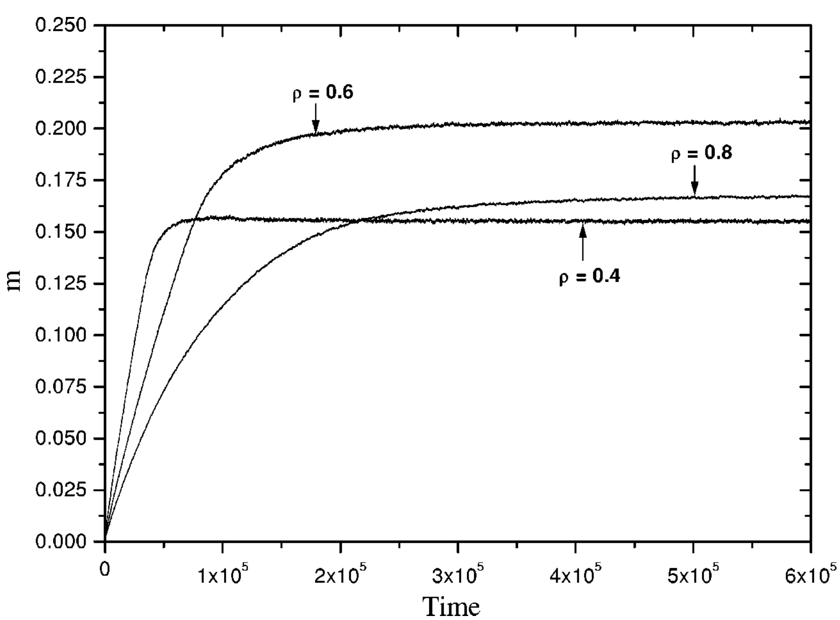

FIG. 2. Time evolutions of m, when starting from the megajam initial configuration, for several values of density ρ ($p=0.005$, $v_{\max}=5$, and $L=1000$).

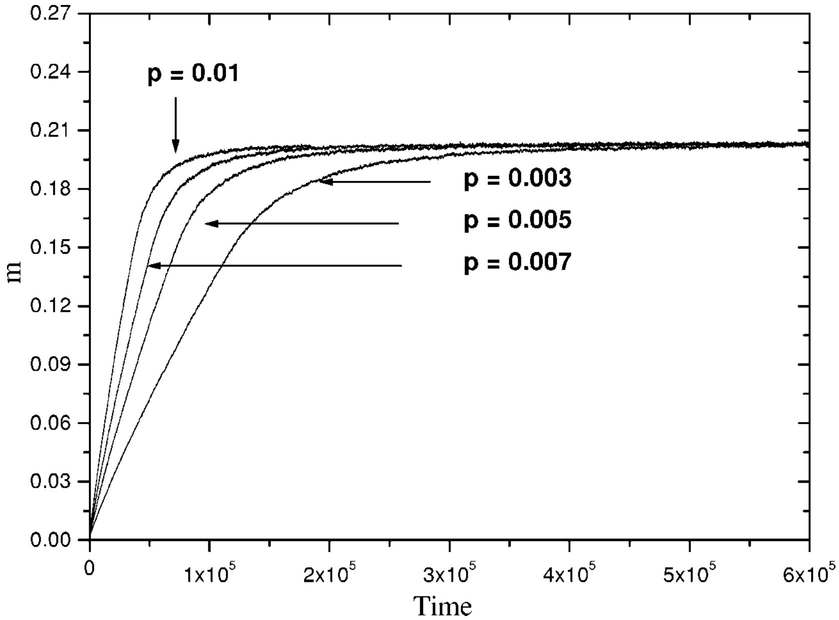

FIG. 3. Time evolutions of m, when starting from the megajam initial configuration, for several values of randomization p ($\rho=0.6$, $v_{\max}=5$, and $L=1000$).

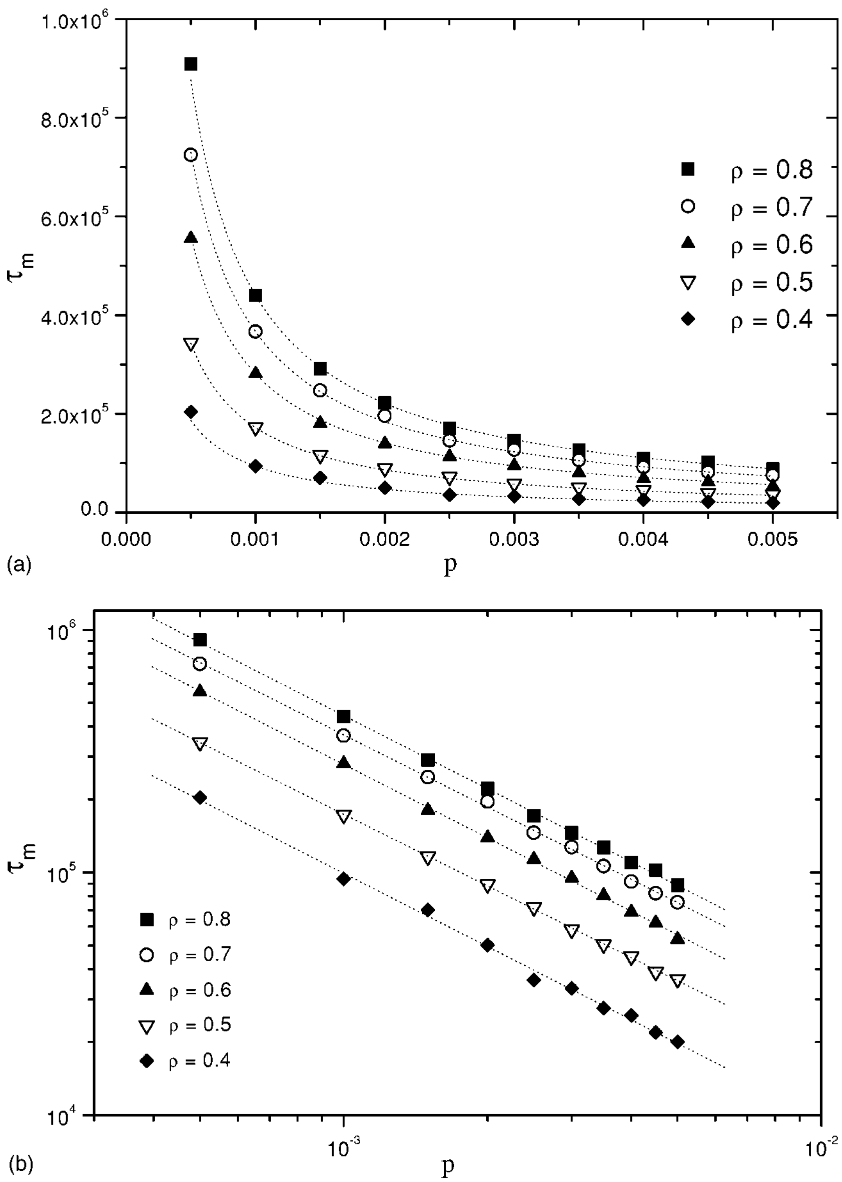

FIG. 4. (a) Variations of the relaxation time τ_m near the limit $p\rightarrow 0$, for several fixed densities. (b) The log-log plot of the relaxation time τ_m ($v_{\max}=5$ and $L=1000$).

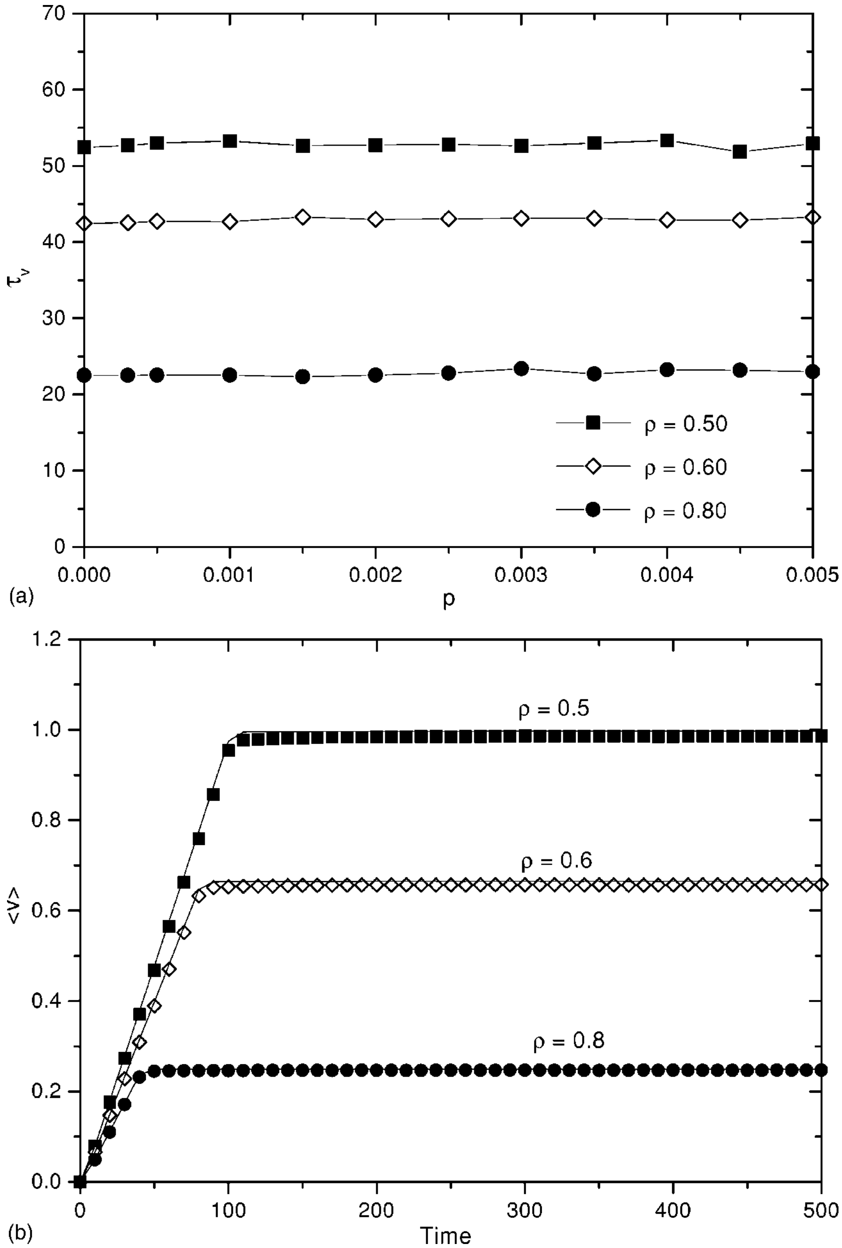

FIG. 5. (a) Variations of the relaxation time τ_v near the limit $p \to 0$, for several fixed densities. (b) Time evolutions of $\langle v \rangle$ for several fixed densities, when starting from the megajam initial configuration. $p=0.0005$ (straight lines) and $p=0.005$ (scatters) ($v_{\max}=5$ and $L=1000$).

$$P(X) \sim \int \int \cdots \int p dt_1 p dt_2 \cdots p dt_n \times e^{-pt_1} e^{-p(t_2-t_1)} \cdots e^{-p(t_n-t_{n-1})}. \quad (3)$$

$P(X)$ is the probability of some event X formed by a succession of transitions of different microscopic states $S_i (i=1,2,\ldots,n)$. For example, one can imagine that the evolution of the system is given as follows: within the time interval $[0,t_1]$ the system is in the state S_1 and then transits to state S_2 at time t_1. The system remains in state S_2 within the time interval $[t_1,t_2]$ and then transits to state S_3 at time t_2, and so on. In Eq. (3), pdt_i represents the probability that the system transits at time t_i, and the probability that the system remains in some state (does not transit) within the time interval $[t_i,t_{i+1}]$ is of the form $e^{-p(t_{i+1}-t_i)}$. It is easy to see from Eq. (3) that $P(X)$ is invariant under the following transformations:

$$p \to \Lambda p \quad \text{and } t \to t/\Lambda, \quad (4)$$

where $\Lambda p < 1$ and Λ is some real constant. Equation (4) implies that the relaxation time τ is of the form

$$\tau \sim \frac{1}{p}. \quad (5)$$

Consequently, this simple theoretical argument predicts that the relaxation time diverges as $p \to 0$ with dynamical exponent $\beta=1$.

To study numerically the relaxation time corresponding to an observable A, we shall use the nonlinear relaxation function [13],

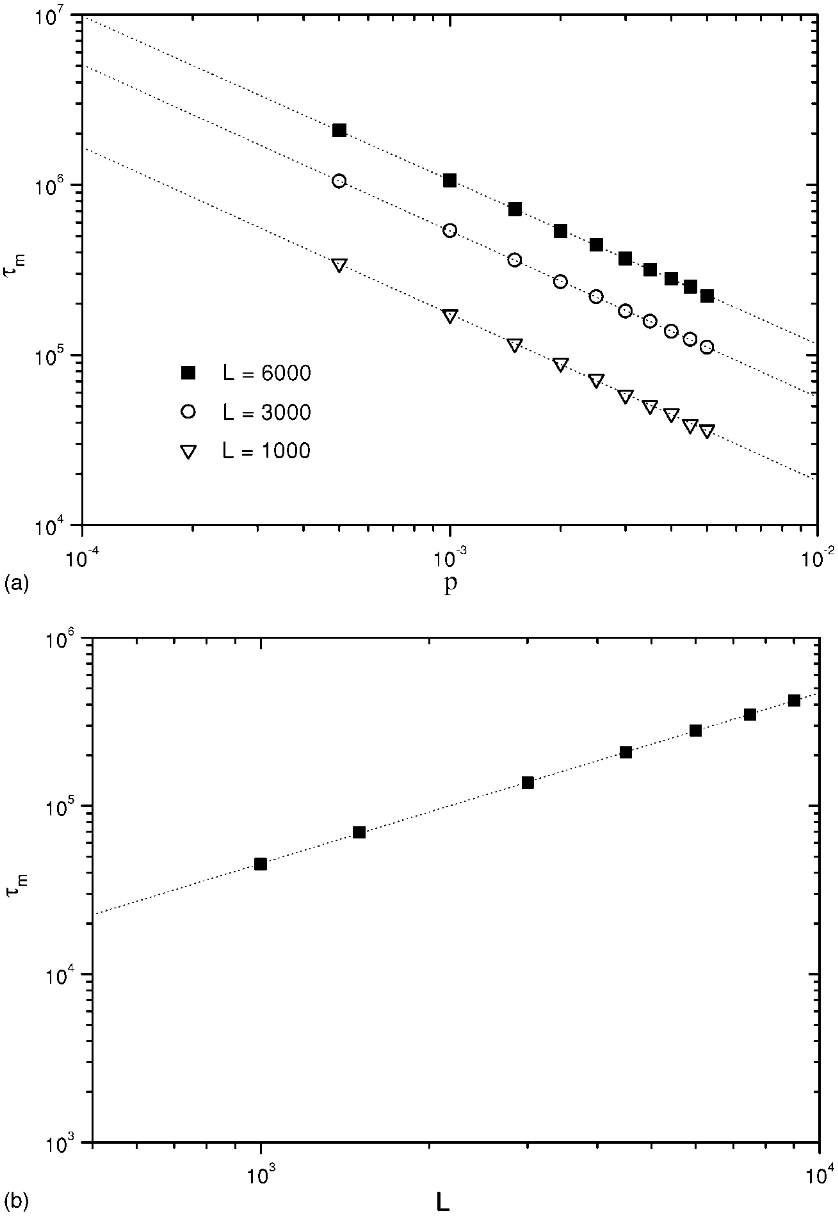

FIG. 6. (a) Variations of the relaxation time τ_m near the limit $p \to 0$ for different lattice size L. (b) Variations of the relaxation time τ_m with size L for $p=0.004$ ($\rho=0.5$ and $v_{\max}=5$).

$$\phi(t) = [A(t) - A(\infty)]/[A(0) - A(\infty)]. \tag{6}$$

The corresponding nonlinear relaxation time

$$\tau = \int_0^\infty \phi(t) dt. \tag{7}$$

The condition that the system is well equilibrated is

$$t_{M_0} \gg \tau, \tag{8}$$

where M_0 is the number of Monte Carlo steps that have to be excluded in the averaging of the observable A. Equation (8) must hold for all quantities A, and hence one must focus on the slowest relaxing quantity to get reliable results. In Fig. 4(a), we plotted the variations of the relaxation time τ_m of the observable m near the limit $p \to 0$, for several fixed densities. As a result, the relaxation time is found to diverge as $p \to 0$ for higher densities. Moreover, as the density of cars is increased, the relaxation time is enhanced. From Fig. 4(b), we see that the relaxation time τ_m follows a power-law behavior of the form

$$\tau \propto p^{-\beta}. \tag{9}$$

Except for some minor fluctuations, the dynamic exponent z remains unchanged when varying the density. For example, $\beta \approx 1.002 \pm 0.001$ for $\rho=0.8$, $\beta \approx 1.004 \pm 0.01$ for $\rho=0.6$, and $\beta \approx 0.981 \pm 0.006$ for $\rho=0.5$.

Let us come back to the other observable $\langle v \rangle(t)$. We see from Fig. 5(a) that the corresponding relaxation time τ_v does not vary when $p \to 0$. If we plot the time evolution of $\langle v \rangle$ for different values of the randomization p [see Fig. 5(b)], we observe that the curves are confounded. Hence, no power-law form is observed if one uses the observable $\langle v \rangle$. In contrast to τ_m, as the density of cars is increased, the relaxation time τ_v is diminished. This shows clearly that the mean ve-

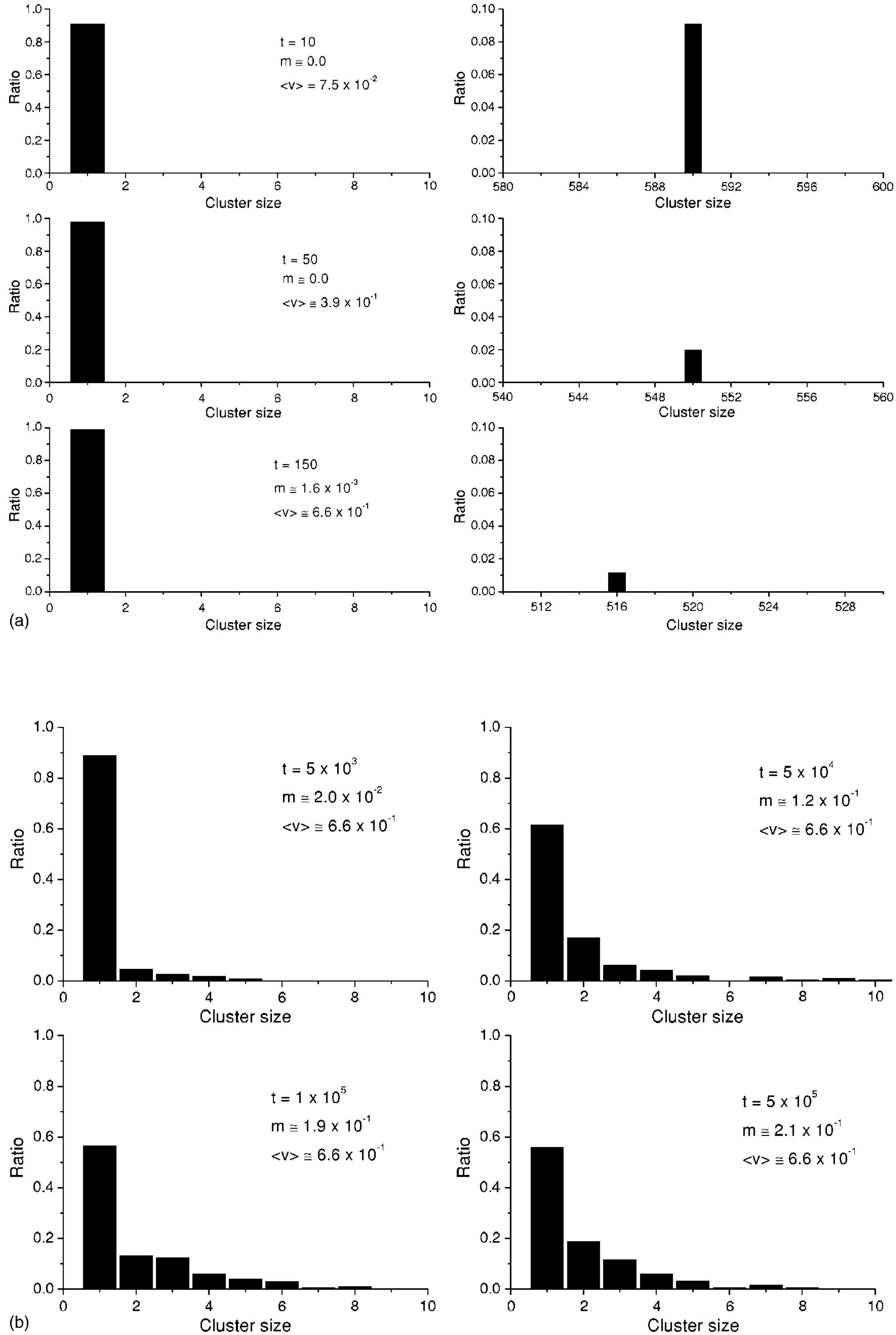

FIG. 7. Cluster size distributions at different time steps. The right side figures of (a) represent distributions of clusters with large sizes ($\rho=0.6$, $p=0.005$, $v_{\max}=5$, and $L=1000$).

locity is not the appropriate observable for studying the dynamics of the traffic flow models.

To study the finite-size effect on the relaxation dynamics, we compute the relaxation time τ_m for different lattice sizes. The results are given in Fig. 6(a), which shows that the power-law behavior exists for any lattice size L. Moreover, we find that the dynamical exponent β is the same for all size L. Yet, for a given randomization p, the finite-size scaling form

$$\tau_m \propto L^z \tag{10}$$

is found [Fig. 6(b)]. We find $z=1.014\pm0.004$, which is compatible with the result found for the deterministic model where an exponent $z=1$ was obtained [14].

To follow the general trend of the cars from the time $t=0$ when all cars are in the megajam, up to the time t_{sat} when the system is saturated, we examine the distribution of the cluster sizes at different time steps [see Figs. 7(a) and 7(b)]. The cluster means here a string of successive stopped cars, i.e., we are considering only compact jams. At earlier time, the system contains only two sizes which correspond to a long cluster and some a few (n_e) escaping cars moving with velocity close to $v_{\max}-p$. No "go and stop" cars exist at this time and the mean velocity is equal to $n_e(v_{\max}-p)$. As t increases, the size of the long cluster decreases and the number of escaping cars increases again. This leads to an increase of the mean velocity, as is shown from Fig. 7(a). $\langle v\rangle(t)$ is saturated at t_0, a time when the number n_e becomes equal to $(L-N)/(v_{\max}-p)$. The mean velocity $\langle v\rangle$ is then equal to $(L-N)/N$. At times longer than t_0, the long cluster disappears whereas clusters with small sizes appear in the lattice, leading therefore to an increase of the density of "go and stop" cars. As Fig. 7(b) shows, the density of "go and stop" cars is saturated at time t_{sat} when the distribution of cluster sizes decays exponentially with the cluster size [12]. Thus, it is claimed that within the interval time $[0,t_0]$, both $\langle v\rangle$ and m evolve with time. For $t>t_0$, m increases with time but $\langle v\rangle(t)$ is saturated. The saturation of m (of the system) is attained at time $t_{sat}\gg t_0$.

Finally, it is important to note that the case where $p\rightarrow0$ is completely different from $p=0$. While the relaxation time diverges in the former case, the system in the second case relaxes immediately after a few time steps (see Fig. 8). The steady states in the DNS model depend on the initial configuration even if the average velocity is unique. However, different initial configurations may lead to different values of the density of "go and stop" cars [15]. In contrast, the steady state of the NDNS model is unique, leading therefore to the uniqueness of both $\langle v\rangle$ and m.

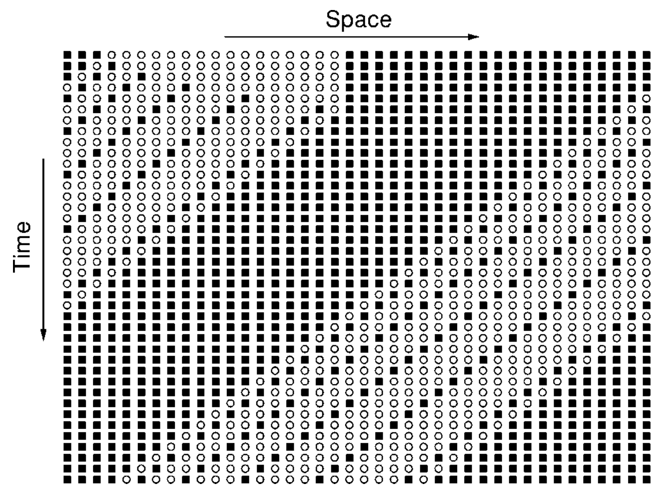

FIG. 8. Spatio-temporal patterns of evolution of cars for $p=0$ and $v_{\max}=5$, starting from a megajam initial configuration. The system size $L=40$ and the number of cars is $N=24$. After exactly six time steps, the steady state is reached where $\langle v\rangle=16/24$ and $m=1/24$.

In summary, the dynamics of the NDNS model in congested traffic could be well understood by studying the time evolutions of the density of "go and stop" cars. As p decreases, the relaxation time τ of the system increases and diverges at the limit $p\rightarrow0$. Therefore, the limit $p\rightarrow0$ should correspond to a critical point of the model. The critical behavior observed is characterized by a dynamical exponent β ($\tau\propto p^{-\beta}$, $\beta=1$).

We thank Joachim Krug for useful discussions and remarks. We greatly acknowledge Deepak Dhar, who contributed immensely to the elaboration of the theoretical argument of the relaxation time developed in this article.

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